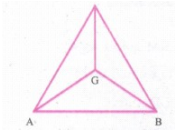
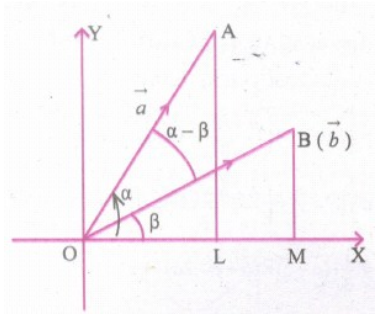
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Answers Key:

5 Marks

1. 

Let the position vector of the vertices of ΔABC be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.
Since G is the centroid of ΔABC, $\vec{OG} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$
Are of ΔGAB, = $|\vec{AB} \times \vec{AG}| = |(\vec{OB} - \vec{OA}) \times (\vec{OG} - \vec{OA})|$
= $\left|(\vec{b} - \vec{a}) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3} - \vec{a}\right)\right|$
= $\left|(\vec{b} - \vec{a}) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}-3\vec{a}}{3}\right)\right|$
= $\left|(\vec{b} - \vec{a}) \times \left(\frac{\vec{b}+\vec{c}+2\vec{a}}{3}\right)\right|$
= $\frac{1}{3}|(\vec{b} - \vec{a}) \times (\vec{a} + \vec{0} - 2\vec{a})|$
= $\frac{1}{3}|\vec{b} \times \vec{b} + \vec{b} \times \vec{c} - 2\vec{b} \times \vec{a} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} + 2\vec{a} \times \vec{a}|$
[∵ cross product is distributive]
= $\frac{1}{3}|\vec{b} \times \vec{c} + 2\vec{a} \times \vec{b} - \vec{a} \times \vec{b} + \vec{c} \times \vec{a}|$
= $\frac{1}{3}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$ $[\because \vec{b} \times \vec{b} = \vec{0}, \vec{a} \times \vec{a} = \vec{0}, .(1)\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}]$
Area of ΔGAC = $|\vec{CA} \times \vec{AG}|$
= $|(\vec{OA} - \vec{OC}) \times (\vec{OG} - \vec{OA})|$
$\left|(\vec{a} - \vec{c}) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3} - \vec{a}\right)\right|$
= $\left|(\vec{c} - \vec{a}) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}-3\vec{a}}{3}\right)\right|$
= $\frac{1}{3}|(\vec{a} - \vec{c}) \times (\vec{b} + \vec{c} - 2\vec{a})|$
= $\frac{1}{3}|\vec{a} \times \vec{b} + \vec{a} \times \vec{c} - 2\vec{a} \times \vec{a} - \vec{c} \times \vec{b} - \vec{c} \times \vec{c} + 2\vec{c} \times \vec{a}|$
= $\frac{1}{3}|\vec{a} \times \vec{b} - \vec{c} \times \vec{a} + \vec{b} \times \vec{c} + 2\vec{c} \times \vec{a}|$
= $\frac{1}{3}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$
Also area of ΔGBC = $|\vec{BC} \times \vec{BG}|$
= $|\vec{OC} - \vec{OB}) \times (\vec{OG} - \vec{OB})|$
= $\left|(\vec{c} - \vec{a}) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}-\vec{b}}{3}\right)\right|$
= $\left|(\vec{c} - \vec{a}) \times \left(\frac{\vec{a}+\vec{b}-2\vec{a}}{3}\right)\right|$
$\frac{1}{3}|(\vec{c} - \vec{b}) \times (\vec{a} + \vec{c} - 2\vec{b})|$
= $\frac{1}{3}|\vec{c} \times \vec{a} + \vec{c} \times \vec{c} - 2\vec{c} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{c} + 2\vec{b} \times \vec{b}|$
= $\frac{1}{3}|\vec{c} \times \vec{a} + 2\vec{b} \times \vec{c} + \vec{a} \times \vec{b} - \vec{b} \times \vec{c}|$
$\frac{1}{3}|\vec{c} \times \vec{a} + \vec{b} \times \vec{c} + \vec{a} \times \vec{b}|$
From (1), (2) and (3),
Area of ΔGAB = Area of ΔGAC = Area of ΔGBC
= $\frac{1}{3}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$
= $\frac{1}{3}$ Area of ΔABC.

2. 

Let $\hat{a} = \vec{OA}$ and $\hat{b} = \vec{OB}$ be the unit vectors and which make angles α, β respectively with positive x-axis Draw AL and BM ⊥ to axis
Then $|\vec{OL}| = |\vec{OA}|$ = cos α = cos α
$|\vec{LA}| = |\vec{OB}|$ sin α = sin α
[∵ ΔOAL, sin α = $\frac{opp}{hyp} cos\alpha = \frac{adj}{hyp}$]
$\therefore \vec{OL} = |\vec{OL}|\hat{i} = cos\alpha\hat{i}$
$\vec{LA} = |\vec{LA}|\hat{j} = sin\ \alpha\hat{j}$
$\therefore \hat{a} = \vec{OA} = \vec{OL} + \vec{LA}$ [Using Δ law of addition]

$\hat{a} = cos\alpha\hat{i} + sin\alpha\hat{j}$

Similarly $\hat{b} = cos\alpha\hat{i} + sin\alpha\hat{j}$ ...(2)

The angle between $\hat{a}$ and $\hat{b}$ is (α–β)

$\therefore \hat{a}.\hat{b} = |\vec{a}||\vec{b}|$cos(α–β) = cos(α–β)..(3)

[$\because |\vec{a}| = |b|$ = 1]

Also $\vec{a}.\vec{b} = (cos\alpha\hat{i} + sin\alpha\hat{j})(cos\beta\hat{i} + sin\beta\hat{j})$

$\hat{a}\hat{b}$ = cosα cosβ + sinα sinβ....(4)

From (3) and (4), we get

cos(α–β) = cos α cos β + sin α sin β

3. Let $\cos^{-1}x = \alpha$ and cos-1 y = β.

Then, x = cosα and y cos =β

$\cos^{-1}x + x + \cos^{-1} y + \cos^{-1}x = \pi$ gives $\alpha+\beta = \pi - \cos^{-1}z$.

Now, cos($\alpha+\beta$) = cosαcosβ-sinαsinβ = xy-$\sqrt{1-x^2}\sqrt{1-y^2}$

-cos(cos^{-1} z) = xy$\sqrt{1-x^2}\sqrt{1-y^2}$

so, $-z = xy - \sqrt{1-x^2}\sqrt{1-y^2}$, Which gives -xy - z = -$\sqrt{1-x^2}\sqrt{1-y^2}$

Squaring on both sides and simplifying, we get $x^2 + y^2 + z^2 + 2xyz = 1$.

4. Now, $tan^{-1}\left(\frac{x-1}{x-2}\right) + tan^{-1}\left(\frac{x+1}{x+2}\right) = tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$

Thus, $\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}$ = 1, which on simplification gives $2x^2-4 = -3$

Thus, $x^2 = \frac{1}{2}$gives x = $\pm\frac{1}{\sqrt{2}}$

5. The required line passes through (1, 2, –3). So, the position vector of the point is $\hat{i} + 2\hat{j} - 3\hat{k}$.

Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 5\hat{j} - 7\hat{k}$. Then, we have

(i) vector equation of the required straight line in parametric form is $\vec{r} = \vec{a} + t\vec{b}$, t ∈ R

Therefore, $\vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + t(4\hat{i} + 5\hat{j} - 7\hat{k})$, t ∈ R

(ii) vector equation of the required straight line in non-parametric form is $(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$

Therefore, $(\vec{r} - (\hat{i} + 2\hat{j} - 3\hat{k})) \times t(4\hat{i} + 5\hat{j} - 7\hat{k}) = \vec{0}$

(iii) Cartesian equations of the required line are $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

Here, (x_1, y_1, z_1) = (1, 2, –3) and direction ratios of the required line are proportional to 4, 5, –7. Therefore, Cartesian equations of the straight line are

$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+3}{-7}$

6. Given point is $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}$

and parallel vector is $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

$\therefore$ The parametric form of vector equation of a line passing through $\vec{a}$ and parallel to $\vec{b}$ is

$\vec{r} = \vec{a} + \vec{b}$

$\vec{r} = (-\hat{i} + 2\hat{j} + \hat{k}) + t(\hat{i} - 2\hat{j} + \hat{k}), k \in R$ (1)

Given line is

$\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) + t(\hat{i} - 2\hat{j} + \hat{k}), k \in R$ (2)

$\therefore \vec{c} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\therefore \vec{c} - \vec{a} = (2\hat{i} + 3\hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + \hat{k})$

$= (3\hat{i} + \hat{j} - 2\hat{k})$

$(\vec{c} - \vec{a}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix}$

$= \hat{i}(1 - 4) - \hat{j}(3 + 2) + \hat{k}(-6 - 1)$

$= -3\hat{i} - 5\hat{j} - 7\hat{k}$

$|(\vec{c} - \vec{a}) \times \vec{b}| = \sqrt{(-3)^2 + (-5)^2 + (-7)^2}$

$= \sqrt{9 + 25 + 49} = \sqrt{83}$

$|\vec{b}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$\therefore$ Distance between the parallel lines

$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|} = \frac{\sqrt{83}}{\sqrt{6}}$ units

7. The plane passes through two points

$\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

The straight line passing through the points

(2, 1, –3) and (-1, 5, -8) is

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$.

$\Rightarrow \frac{x-2}{-1-2} = \frac{y-1}{5-1} = \frac{z+3}{-8+3}$

$\Rightarrow \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z+3}{-5}$

Hence the required plane is parallel to the vector

$\vec{c} = -3\hat{i} + 4\hat{j} - 5\hat{k}$

The parametric form of vector equation of the plane passing through two points $\vec{a}, \vec{b}$ parallel to a vector$\vec{c}$ *is* $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}, s, t \in R$,

$\vec{r}.2\hat{i} + 2\hat{j} + \hat{k} + s(-\hat{i} - 4\hat{j} + 2\hat{k}) + t(3\hat{i} - 4\hat{j} + 5\hat{k})$ s, t ∈ R

Cartesian form of the plane passing through two points and parallel to a vector is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

[$\because$(x_1,y_1, z_1) is (2, 2, 1), (x_2, y_2, z_2) is (-1, -2, 3) & (c_1, c_2, c_3) is (-3, 4 -5)

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$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$

$\Rightarrow$ (x - 2)(20 - 8) - (y- 2)(5+ 6) + (z-1)(-4 -12) = 0

$\Rightarrow$ (x - 2)(12) - (y - 2)(11) + (z - 1)(-16) = 0

$\Rightarrow$ 12x- 24 -11y + 22 -16z + 16 = 0

$\Rightarrow$ 12x-11y-16z+14 = 0

8. $tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

Let a = tan 0

Now, $tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right] = tan\left[\frac{1}{2}sin^{-1}\left(\frac{2tan\theta}{1+tan^2\theta}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)\right]$

$tan\left[\frac{1}{2}sin^{-1}(sin2\theta)+\frac{1}{2}cos^{1}(cos2\theta)\right] = tan[2\theta] = \frac{2tan\theta}{1-tan^2\theta} = \frac{2a}{1-a^2}$

9. When x = 4, dx = 4.1-4 = 0.1

10. $f_y = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right) = \frac{\frac{-x}{y^2}}{\frac{x^2+y^2}{y^2}}$

$= \frac{-x}{x^2+y^2}$

$f_x = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right) = \frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}}$

$= \frac{y}{x^2+y^2}$

$f_{xy} = \frac{\partial}{\partial x}(f_y)$

$f_{xy} = -\left[\frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}\right]$

$= -\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right]$

$= -\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$

$= \frac{x^2-y^2}{(x^2+y^2)^2}$ ...(1)

$f_{xy} = \frac{\partial}{\partial y}(f_x)$

$f_{xy} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$= \frac{x^2-y^2}{(x^2+y^2)^2}$ ...(2)

$\therefore$ From (1) and (2), $f_{xy} = f_{yz}$

11. We shall follow the tree diagram to calculate

So first we need to find $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$, $\frac{dx}{dt}$ and $\frac{dx}{dt}$.

Now, $\frac{\partial g}{\partial x}$ = 2x-y + cos(x + y), $\frac{\partial g}{\partial x}$ = -x+cos(x + y), $\frac{dx}{dt} = 3e^{3t}$ and $\frac{dx}{dt}$ = 2t.

Thus, $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}$

= (2x - y + cos(x + y)) $3e^{3t}$ + (-x + cos(x + y))(2t)

= $(2e^{3t} - t^2 + cos(e^{3t} - t^2))3e^{3t}$ +($-e^{3t}$ + $cos(e^{3t} - t^2)$)(2t)

= $6e^{6t} - 3t^2 e^{3t} + 3e^{3t} cos(e^{3t} - t^2) - 2te^{3t} + 2t cos(e^{3t} - t^2)$

Also, some times our W(x, y) will be such that x = x(s, t) , and y = y(s, t) where s, t $\in$ R. Then W can be considered as a function that depends on s and t. If x, y both have partial derivatives with respect to s, t and W has partial derivatives with respect to x and y, then we can calculate the partial derivatives of W with respect to s and t using the following theorem.

12. Given w (x, y, z) = $x^2 + y^2 + y^2$,

$x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

$\frac{\partial u}{\partial x}$ = 2x; $\frac{\partial u}{\partial y}$ = 2y; $\frac{\partial u}{\partial z}$ = 2z

$\frac{\partial u}{\partial x} = 2e^t$

$\frac{\partial u}{\partial y} = 2e^t \sin t$

$\frac{\partial u}{\partial z} = 2e^t \cos t$

$\frac{dx}{dt} = e^t$

$\frac{dy}{dt} = e^t$ cas t + sin t e^t

$\Rightarrow \frac{dz}{dt}$ = et (- sin t) + cos t et

By chain rule

$\frac{dw}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{dw}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$

$\therefore \frac{dw}{dt}$ = $2e^t(e^t)$ + $2e^t$ sin t (e^t cos t +sin t e^t) - e^t sin t + $2e^t$ cas t (e^t cos t - e^t sin t)

$\therefore \frac{dw}{dt} = e^{2t}\left[2 + 2\sin t\cos t + 2\sin^2 t - 2\sin t\cos t + 2\cos^2 t\right]$

= e^{2t} [2 + 2] = $4e^{2t}$

13. Given U (x, y) = e^x sin y ; x = st^2 ; y = s^2t

$\frac{\partial U}{\partial x}$ = e^x sin y ; $\frac{\partial U}{\partial y}$ = e^x cos y

$\frac{\partial U}{\partial x} = e^{st^2}$ sin (s^2t)

$\frac{\partial U}{\partial y} = e^{st^2}$ cos (s^2t)

$\frac{dx}{dt}$ = 2st; $\frac{dy}{dt} = s^2$

$\frac{dx}{ds} = t^2$; $\frac{dy}{ds}$ = 2 st

By chain rule

$\frac{dU}{ds} = \frac{\partial U}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial U}{\partial y}\cdot\frac{dy}{ds}$

= e^{st^2}. sin (s^2t) (t^2) + e^{st^2} cos(s^2t).(2^{st})

$\therefore \left(\frac{\partial U}{\partial s}\right)_{(s=t=1)}$ = e^1 sin (1) + $2e^1$ cos (1)

= e [sin (1) + 2 cos (1)] and

$\frac{dU}{dt} = \frac{\partial u}{\partial z}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$

= e^{st^2} . sin (s^2t)(2^{st}) + e^{st^2} cos (s^2t). (s^2)

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$\therefore \left(\frac{\partial U}{\partial t}\right)_{(s=t=1)}$ = $2e^1 \sin(1) + e^1 \cos(1)$

= e [2 sin (1) + cos (1)]

14. $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}; \overrightarrow{OB} = 4\hat{i} + \hat{j} + 5\hat{k}; \overrightarrow{OC} = 7\hat{i} + 7\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\hat{i} + \hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$\overrightarrow{AB} = 3\hat{i} - \hat{j} + 2\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 6\hat{i} - 2\hat{j} + 4\hat{k}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 6 & -2 & 4 \end{vmatrix} = \vec{0}$

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. But they have the point A as a common point.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are along same line

$\therefore$ A, B, C are collinear.

3 Marks

15. With usual notations in triangle ABC, we have $\vec{BC} = \vec{a}, \vec{CA} = \vec{b}\ \vec{AB} = \vec{c}$.

Then $\left|\vec{BC}\right| = a$, $\left|\vec{CA}\right| = b$, $\left|\vec{AB}\right| = c$, and $\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$

So, $\vec{BC} = -\vec{CA} - \vec{AB}$

Then applying dot product, we get

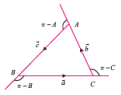
$\vec{BC}.\vec{BC} = (-\vec{CA} - \vec{AB}).(-\vec{CA} - \vec{AB})$

$\Rightarrow \left|\vec{BC}\right|^2 = \left|\vec{CA}\right|^2 + \left|\vec{AB}\right|^2 + 2\vec{CA}.\vec{AB}$

$\Rightarrow a^2 = b^2 + c^2 + 2bc \cos(\pi - A)$

$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$.

The results (ii) and (iii) are proved in a similar way.



16. Given $f(x) = sin^{-1}\left(\frac{x^2+1}{2x}\right) \leq 1$

We know that the domain of $\sin^{-1}(x)$ is [-1, 1]

$\Rightarrow -1 \leq \frac{x^2+1}{2x} \leq 1$

Consider $\Rightarrow -1 \leq \frac{x^2+1}{2x}$

$\Rightarrow 0 \leq \frac{x^2+1}{2x} + 1$

$\Rightarrow \frac{x^2+1+2x}{2x} \geq 0$

$\Rightarrow \frac{(x+1)^2}{2x} \geq 0$

$\Rightarrow$ x = -1 and x < 0

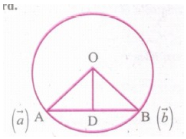
Consider $\frac{x^2+1}{2x} \leq 1$

$\Rightarrow \frac{x^2+1}{2x} - 1 \leq 0$

$\Rightarrow \frac{x^2-2x+1}{2x} \leq 0$

$\Rightarrow \frac{(x-1)^2}{2x} \leq 0$

From (1) and (2) Domain {-1, 1}



17.

Let the position vectors of the parts A and B on the circle lie $\vec{a}$ and $\vec{b}$ respectively.

Since O is the centre of the circle

$|\vec{OA}| = |\vec{OB}| \Rightarrow |\vec{a}| = |\vec{b}|$ ....(1)

Also D is the mid-point of AB,

$\Rightarrow \vec{OD} = \frac{\vec{a}+\vec{b}}{2}$ (mid-point formula)

$\left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{OB} - \vec{OA})$

$= \left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{Ob} - \vec{Oa})$

$= \frac{1}{2}\left[|\vec{b}|^2 - |\vec{a}|^2\right]$

$= \left[\because (\vec{a} + \vec{b}).(\vec{b} - \vec{a}) = |\vec{b}|^2 - |\vec{a}|^2\right]$

$= \frac{1}{2}\left[|\vec{b}|^2 - |\vec{b}|^2\right]$ (using (1))

$= \frac{1}{2}(0) = 0$

$\Rightarrow \vec{OD}.\vec{AB} = 0 \Rightarrow \vec{OD} \perp \vec{AB}$

Hence, if a line is drawn from the centre of a to the mid-point of a chord, then that line is perpendicular to the chord.

18. Let $\sin^{-1} x = \theta$.

Then, x = $\sin\theta$ and $-1 \leq x \leq 1$

Now, $\tan(\sin^{-1} x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{\sqrt{1-x^2}}, |x| < 1$

19.

Cartesian equation of straight line passing through two points (2, 1,4) and (a - 1,4, -1) is

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

[$\because$ (x_1, y_1, z_1) = (2, 1, 4) (x_2, y_2, z_2) = (a, -1, 4, -1)]

$\Rightarrow \frac{x-2}{a-1-2} = \frac{y-1}{4-1} = \frac{z-4}{-1-4}$

$\Rightarrow \frac{x-2}{a-3} = \frac{y-1}{3} = \frac{z-4}{-5}$

Similarly, Cartesian equation of straight liens passing through two points (0, 2, b - 1) and (5, 3, -2) is

$\frac{x-5}{5-0} = \frac{y-3}{2-3} = \frac{z+3}{b-1+2}$

[$\because$ (x_1,y_1,z_1)=(5,3,-2) (x_2,y_2,z_2) is (0,2,b)...(1)

$\Rightarrow \frac{x-5}{-5} = \frac{y-3}{-1} = \frac{z+2}{b+1}$

(1) and (2) are parallel if their direction cosines are equal

$\therefore$ Direction ratios ofline (1) are a - 3, 3, -5 ..... (3)

Direction ratios of line (2) are -5, -1, b + 1 ..... (4)

To make the direction ratios equal, multiply (4) by-3.

$\therefore$ (4) $\longrightarrow$ +15, 3, -3b-3

(3) $\longrightarrow$ a-3, 3, -5

$\therefore$ a-3 = +5 $\Rightarrow$ a = 15+3 = 18

3 = 3

-3b-3 = -15 $\Rightarrow$ -3b = -5+3 = -2

$\Rightarrow$ b = $\frac{-2}{-3} = \frac{2}{3}$

$\therefore$ a = 18 and b = $\frac{2}{3}$.



20.

Let $cos^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow cos\theta = \frac{4}{5}$

$\therefore tan\theta = \frac{opp}{adj} = \frac{3}{4}$

$\Rightarrow \theta = tan^{-1}\left(\frac{3}{4}\right)$

LHS = $cos^{-1}\left(\frac{4}{5}\right) + tan^{-1}\left(\frac{3}{5}\right)$

= $tan^{-1}\left(\frac{3}{4}\right) + tan^{-1}\left(\frac{3}{5}\right)$

$\left[\because tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

= $tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)}\right)$

= $tan^{-1}\left(\frac{\frac{15+12}{20}}{\frac{20-9}{20}}\right) = tan^{-1}\left(\frac{27}{20} \times \frac{20}{11}\right)$

= $tan^{-1}\left(\frac{27}{11}\right) = RHS$

Hence proved

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21. Given h (x, y, z) = x sin (xy) + z^2x

$\frac{\partial h}{\partial x} = x.cos(x,y).\frac{\partial}{\partial x}(xy) + sin(xy)(1) + z^2(1)$

= x cos (xy) (y)(1) + sin (xy) + z^2

= xy cos (xy) + sin (xy) + z^2

$\therefore \left(\frac{\partial h}{\partial z}\right)_{(2,\frac{\pi}{4},1)} = 2\left(\frac{\pi}{4}\right)cos\left(2\frac{\pi}{4}\right) + sin\left(2\frac{\pi}{4}(1)\right) + 1^2$

$= \frac{\pi}{2}cos\left(\frac{\pi}{2}\right) + sin\left(\frac{\pi}{2}\right) + 1$

$= \frac{\pi}{2}(0) + 1 + 1 = 2$

$\frac{\partial h}{\partial y} = x.cos(xy).\frac{\partial h}{\partial y}(xy) + 0$

= x cos (xy) x(1)

= x^2 cos (xy)

$\therefore \left(\frac{\partial h}{\partial z}\right)_{(2,\frac{\pi}{4},1)} = 2^2cos\left(2\frac{\pi}{4}\right)$

= $4cos\left(\frac{\pi}{2}\right)$

= 4(0) = (0)

$\left(\frac{\partial h}{\partial z}\right) = 0 + x(2z) = 2xz$

$\therefore \left(\frac{\partial h}{\partial x}\right)_{(2,\frac{\pi}{4},1)} = 2(2)(1) = 4$

22. Given u (x, y) = $\frac{x^2+y^2}{\sqrt{x+y}}$

$u(\lambda x, \lambda y) = \frac{\lambda^2x^2+\lambda^2y^2}{\sqrt{\lambda x+\lambda y}}$

= $\frac{\lambda^2(x^2+y^2)}{\sqrt{\lambda}(\sqrt{x+y})}$

= $\lambda^{2-\frac{1}{2}}u(x,y)$

= $\lambda^{\frac{3}{2}}u(x,y)$

$\therefore$ u (x, y) is a homogeneous function of degree $\frac{3}{2}$

$\therefore$ By Euler's theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = n.u $\asymp x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

Hence, proved.

23. $\frac{5}{18}$

24. L.H.S = $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$

$= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} + (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a} + (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$

$= \vec{0}$ = R.H.S

2 Marks

25. Let $\text{cosec}^{-1}(-1) = y$. Then, cosec y = -1
Since the range of principal value branch of $y = \text{cosec}^{-1}x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]\backslash\{0\}$ and
cosec $(-\frac{\pi}{2}) = -1$,
We have $y = -\frac{\pi}{2}$.
Note that $-\frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]\backslash\{0\}$.
Thus, the principal value of $\text{cosec}^{-1}(-1)$ is $-\frac{\pi}{2}$

26. Let $\vec{a} = 7\hat{i} + \lambda\hat{j} - 3\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = -3\hat{i} + 7\hat{j} - 5\hat{k}$
$\therefore$ volume of the parallelepiped
$= \vec{a}.(\vec{b} \times \vec{c})$
Given $\vec{a}.(\vec{b} \times \vec{c}) = 90$
$$\Rightarrow \begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90$$
$$\Rightarrow -6\begin{vmatrix} 2 & -1 \\ 7 & 5 \end{vmatrix} - \lambda\begin{vmatrix} 1 & -1 \\ -3 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ -3 & 7 \end{vmatrix} = 90$$
$\Rightarrow$ 7(10+7)-λ(5-3)-3(7+6) = 90
$\Rightarrow$ 7(17)-λ(2)-3(13) = 90
$\Rightarrow$ 119-2λ-39 = 90
$\Rightarrow$ 119-39-90 = 2λ
$\Rightarrow$ -10 = 2λ
$\Rightarrow$ λ = -5

27. Given $\vec{a} = \hat{i} - \hat{k}, \vec{b} = x\hat{i} + \hat{j} + (1-x)\hat{k}, \vec{c} = y\hat{i} + x\hat{j} + (1+x+y)\hat{k}$
$$[\vec{a}, \vec{b}\vec{c}] = \vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$
$$= 1\begin{vmatrix} 1 & 1-x \\ x & 1+x-y \end{vmatrix} + 0 - y\begin{vmatrix} x & y \\ y & x \end{vmatrix}$$
= [(1+x-y)-x(1-x)]-[x^2-y]
$= 1 + \not{x} - \not{y} - \not{x} + \not{x}^2 - \not{x}^x + \not{y}$
= 1
$\therefore [\vec{a}\vec{b}\vec{c}]$ = 1 for all values of x and y
$\therefore [\vec{a}\vec{b}\vec{c}]$ depends on neither x nor y.

28. $y = -sin\left(\frac{1}{3}x\right)$
The amplitude sin x is 1
$\Rightarrow$ amplitude of $-sin\left(\frac{1}{3}x\right)$ is also 1.
The period of $-sin\left(\frac{1}{3}x\right)$ is $\frac{1}{3}x = 2\pi \Rightarrow x = 6\pi$

29. We know that $\sin^{-1}$: [-1, 1] $\rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by
$\sin^{-1}x = y$ if and only if x = sin y for $-1 \leq x \leq$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Thus,
$sin^{-1}\left(sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$, since $-\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

30. Let sin-1(-1) = y where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
Then $sin^{-1}(-1) = y \Rightarrow sin \quad y = -1$
$-1 = sin\left(\frac{-\pi}{2}\right) \Rightarrow y = \frac{-\pi}{2}$ $\left[\because \frac{-\pi}{2} \epsilon \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right]$
$\therefore$ The principal value of $sin^{-1}(-1)\ is\ \frac{-\pi}{2}$

31. Note that g is differentiable and g′(x) = 2x + cos x
Thus dg = (2x + cos x)dx.

32. x = 2 and dx = 0.1
Taking differentials,
df = (2x + 3) dx
Whenx = 2, dx = 0.1
df = (2(2) + 3)(0.1) = 7(0.1) = 0.7

33. $\frac{\partial^2 u}{\partial x \partial y} = e^{xy}\left[3xy + 1 + x^2y^2\right]$

34. Compare the given equation with
$\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$
$\vec{a} = \hat{i} - \hat{j}\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$
$\vec{c} = \hat{i} + \hat{j} - \hat{k}\vec{d} = 2\hat{i} - \hat{j} - \hat{k}$
$\vec{c} - \vec{a} = 2\hat{j} - \hat{k}$
$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$
$= \hat{i}(-1+1) - \hat{j}(-2-2) + \hat{k}(-2-2)$
$= 4\hat{j} - 4\hat{k}$
$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (2\hat{j} - \hat{k}) \cdot (4\hat{j} - 4\hat{k})$
$= 8 + 4 = 12$
$|(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})| = 12$
$|(\vec{b} \times \vec{d})| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

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Distance $= \frac{|(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$

$= \frac{12^3}{4\sqrt{2}} = \frac{3}{\sqrt{2}}$ units

Multiple Choice Question

35. (c) $\frac{\pi}{10}$
36. (d) $tan^{-1}\left(\frac{1}{2}\right)$
37. (c) $\frac{\pi}{6}$
38. (b) $x^2-x-12 = 0$
39. (d) $\frac{x}{\sqrt{1+x^2}}$
40. (c) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$
41. (a) $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$
42. (c) π
43. (a) 0
44. (a) 81
45. (a) $\frac{1}{2}, -2$
46. (a) $\frac{\pi}{3}$
47. (b) 0.4%
48. (b) $\frac{1}{5}$
49. (b) $\frac{1}{(x+1)^2}dx$
50. (b) y - z
51. (a) 7, 11
52. (b) $\frac{\sqrt{3}}{2}$
53. (c) $\vec{c}$ parallel to $\vec{a}$
54. (c) parallel to the line of intersection of the plane containing $\vec{a}$ and $\vec{b}$ and the plane containing $\vec{c}$ and $\vec{d}$

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