St.Anne's Mat.Hr.Sec.School

UNIT TEST

12th Standard 2019 EM

Date : 23-Jul-19

Maths

Reg.No. :

Time : 01:15:00 Hrs

Total Marks : 45

PART-A

5 x 1 = 5

1) A polynomial equation in x of degree n always has

(a) n distinct roots (b) n real roots (c) n imaginary roots (d) at most one root

2) If α,β and γ are the roots of x^3+px^2+qx+r, then $\Sigma\frac{1}{\alpha}$ is

(a) $-\frac{q}{r}$ (b) $\frac{p}{r}$ (c) $\frac{q}{r}$ (d) $-\frac{q}{p}$

3) The number of positive roots of the polynomial $\Sigma_{j=0}^{n} n_{C_r}(-1)^r x^r$ is

(a) 0 **(b) n** (c) < n (d) r

4) If x is real and $\frac{x^2-x+1}{x^2+x+1}$ then

(a) $\frac{1}{3}$
≤k≤ (b) k≥5 (c) k≤0 (d) none

5) If ∝, β, ɤ are the roots of the equation x^3-3x+11=0, then ∝+β+ɤ is __________.

(a) 0 (b) 3 (c) -11 (d) -3

PART-B

5 x 2 = 10

6) If p and q are the roots of the equation $lx^2+nx+n = 0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

Given p,q are the roots of lx^2+nx+n=000

$p + q = \frac{-b}{a} = \frac{-n}{l} \qquad \ldots(1)$

$pq = \frac{c}{a} = \frac{n}{l} \qquad \ldots(2)$

and
$$\left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}}\right)$$

consider
$$\left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}}\right)$$

$$= \frac{p}{q} + \frac{q}{p} + \frac{n}{l} + 2\sqrt{\frac{pq}{qb}} + 2\sqrt{\frac{qn}{pl}} + 2\sqrt{\frac{np}{ql}}$$

$[\because (a + bc)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca]$

$$\Rightarrow \frac{p^2+q^2}{pq} + \frac{n}{l} + 2 + 2\sqrt{\frac{p}{q}.pq} + 2\sqrt{\frac{q}{p}.pq} + 2\sqrt{\frac{p}{q}.pq}$$

$$= \frac{(p+q)^2-2pq}{pq} + \frac{n}{l} + 2 + 2\sqrt{p^2} + 2\sqrt{q^2}$$

$$= \frac{(p+q)^2}{pq} - \frac{2pq}{pq} + \frac{n}{l} + 2 + 2p + 2q$$

$$\Rightarrow \frac{\left(\frac{-n}{l}\right)^2}{\frac{n}{l}} - \cancel{2} + \frac{n}{l} + \cancel{2} + 2(p+q)$$

$$\Rightarrow \frac{n^2}{l^2.\frac{n}{l}} + \frac{n}{l} - \frac{2n}{l} \qquad [\because p + q = \frac{-n}{l}]$$

$$\Rightarrow \frac{n}{l} + \frac{n}{l} - \frac{2n}{l} = 0 \therefore \left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}}\right)^2 = 0$$

Talking square root both sides, we get

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$$

7) Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root.

Given $(\sqrt{5} - \sqrt{3})$ is a root

$\Rightarrow \sqrt{5} + \sqrt{3}$

$\therefore$ Sum of the roots $= \sqrt{5} - \sqrt{3} + \sqrt{5} + \sqrt{3} = 2\sqrt{5}$

Product of the roots

$= (\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

$= (\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3 = 2$

$\therefore$ One of the factor is X^2 -x (sum of the roots) + product of the roots

$\Rightarrow x^2 - 2x\sqrt{5} + 2$

The other factor also will be $x^2 - 2x\sqrt{5} + 2$

$(x^2 - 2x\sqrt{5} + 2)(x^2 + 2x\sqrt{5} + 2) = 0$

$\Rightarrow (x^2 + 2 - 2\sqrt{5}x)(x^2 + 2 + 2\sqrt{5}x) = 0$

$\Rightarrow \left(x^2 + 2\right)^2 - (2\sqrt{5}x)^2 = 0$

[$\because$ $(a + b)(a - b) = a^2 - b^2$]

$\Rightarrow x^4 + 4x^2 + 4(5)x^2 = 0$

$\Rightarrow x^4 + 4x^2 + 4 - 20x^2 = 0$

$\Rightarrow x^4 - 16x^2 + 4 = 0$

8) Solve: $8x^{\frac{3}{2x}} - 8x^{\frac{-3}{2x}}$=63

$8x^{\frac{3}{2x}} - 8x^{\frac{-3}{2x}}$=63

$\Rightarrow 8\left[\left(x^{\frac{1}{2n}}\right)^3 - \left(x^{\frac{-1}{2n}}\right)^3\right] = 63$

Put $x^{\frac{1}{2n}} = y$

$\Rightarrow 8\left(y^2 - \frac{1}{y^3}\right) = 63$

$\Rightarrow y^3 - \frac{1}{y^3} = \frac{63}{8} \Rightarrow \frac{y^6 - 1}{y^3} = \frac{63}{8}$

$\Rightarrow 8y^6 - 8 = 63y^3$

$\Rightarrow 8y^6 - 63y^3 - 8 = 0$

$\Rightarrow 8t^2 - 63t - 8 = 0 \quad [where \quad t = y^3]$

$\Rightarrow (8t - 1)(t - 8) = 0$

$\Rightarrow t = \frac{1}{8}, 8$

Case (i)when $t = 8, \Rightarrow y^3 = 8 \Rightarrow y^2 = 2^3$

$\Rightarrow y = 2$

Case (ii)when $t = \frac{1}{8}, y^3 = \frac{1}{8} \Rightarrow y = \frac{1}{2}$

When $y = 2, x^{\frac{1}{2n}} = 2$

$\Rightarrow x = (2)^{2n} \quad \Rightarrow x = (2^2)^n$

$\Rightarrow x = 4^n$

When $y = \frac{1}{2}, x^{\frac{1}{2n}} = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^{2n}$

$\Rightarrow x = \left(\frac{1}{2^2}\right)^n = \frac{1}{4^n}$

Hence the roots are 4^n.

9) Examine for the rational roots of x^8-3x+1=0

x^8-3x+1=0

Here a_n = 1, a_o = 1

If $\frac{p}{q}$ is a root of the polynomial, then as

(p, q) = 1p is a factor of a_o = 1 and q is a factor of a_n= 1

Since 1 has no factors, the given equation has no rational roots.

10) If sin ∝, cos ∝ are the roots of the equation $ax^2 + bx + c$-0 (c ≠ 0), then prove that $(n + c)^2 - b^2 + c^2$

Sum of the roots = sin ∝ + cos ∝ = $\frac{-b}{a}$

Product of the roots = sin ∝ cos ∝ = $\frac{c}{a}$

Now1= $\cos^2$∝+$\sin^2$∝

= (sin ∝ +cos ∝)^2 -2 sin ∝ cos ∝

$1 = \frac{b^2}{a^2} - \frac{2c}{a} \Rightarrow 1 = \frac{b^2 - 2ac}{a^2}$

$\Rightarrow a^2 = b^2 - 2ac \Rightarrow a^2 + 2ac = b^2$

Addingcsboth sides, $a^2 + 2ac + c^2 = b^2 + c^2$

$\Rightarrow (a+c)^2 = b^2 + c^2$

PART-C

4 x 3 = 12

11) Find the sum of the squares of the roots of $ax^4+bx^3+cx^2+dx+e = 0$.

Let α,β,γ and δ be the roots of $ax^4+bx^3+cx^2+dx+e=0$

$\Sigma_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$,

$\Sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$,

$\Sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$

$\Sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a}$

We have to find $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

Applying the algebraic identity

$(a+b+c+d)^2 \equiv a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$,

we get

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= \left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= \frac{b^2-2ac}{a^2}$.

12) If $2+i$ and $3-\sqrt{2}$ are roots of the equation $x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0$, find all roots.

Since the coefficient of the equations are all rational numbers, $2+i$ and $3-\sqrt{2}$ are roots, we get $2-i$ and $3+\sqrt{2}$ are also roots of the given equation. Thus $(x-(2+i))$, $(x-(2-i))$, $(x-(3-\sqrt{2}))$ and $(x-(3+\sqrt{2}))$ are factors. Thus their product. $((x-(2+i))(x-(2-i))(x-(3-\sqrt{2}))(x-(3+\sqrt{2}))$ is a factor of the given polynomial equation. That is, $(x^2-4x+5)(x^2-6x+7)$ is a factor. Dividing the given polynomial equation by this factor, we get the other factor as (x^2-3x-4) which implies that 4 and −1 are the other two roots. Thus

$2+i, 2-i, 3+\sqrt{2}, -\sqrt{2}, -1$, and 4 are the roots of the given polynomial equation.

13) If the roots of $x^3+px^2+qx+r=0$ are in H.P. prove that $9pqr = 27r^3+2p$.

Let the roots be in H.P. Then, their reciprocals are in A.P. and roots of the equation

$\left(\frac{1}{x}\right)^3 + p\left(\frac{1}{x}\right)^2 + q\left(\frac{1}{x}\right) + r = 0 \Leftrightarrow rx_{3} + qx_{2} + px + 1 = 0$ ....(1)

Since the roots of (1) are in A.P., we can assume them as $\alpha-d, \alpha, \alpha+d$.

Applying the Vieta's formula, we get

$\Sigma_1 = (\alpha-d)+\alpha+(\alpha+d) = -\frac{q}{r} \Rightarrow 3\alpha = -\frac{q}{r} \Rightarrow \alpha = -\frac{q}{3r}$.

But, we note that α is a root of (1). Therefore, we get

$\left(-\frac{q}{3r}\right)^2 + q\left(-\frac{q}{3r}\right)^2 + p\left(-\frac{q}{3r}\right) + 1 = 0 \Rightarrow q_{3} + 3q_{3} - 9pqr + 27r_{2} = 0 \Rightarrow 2q_{3} + 27r_{2}$.

14) Find solution, if any, of the equation $2\cos^2 x - 9\cos x + 4 = 0$

The left hand side of this equation is not a polynomial in x . But it looks like a polynomial. In fact, we can say that this is a polynomial in cos x . However, we can solve the equation (1) by using our knowledge on polynomial equations. If we replace cos x by y , then we get the polynomial equation $2y^2-9y+4=0$ for which 4 and $\frac{1}{2}$ are solutions.

From this we conclude that x must satisfy $\cos x = 4$ or $\cos x = \frac{1}{2}$. But $\cos x = 4$ is never possible, if we take $\cos x = \frac{1}{2}$, in fact, for all $n \in Z$, $x = 2n\pi \pm \frac{\pi}{3}$ are solutions for the given equation (1).

If we repeat the steps by taking the equation $\cos^2 x - 9\cos x + 20 = 0$, we observe that this equation has no solution.

PART-D

4 x 5 = 20

15) If $x^2+2(k+2)x+9k=0$ has equal roots, find k.

Here $\Delta=b^2-4ac=0$ for equal roots. This implies $4(k+2)^2=4(9)k$. This implies k = 4 or 1.

16) Prove that a line cannot intersect a circle at more than two points.

By choosing the coordinate axes suitably, we take the equation of the circle as $x^2+y^2=r^2$ and the equation of the straight line as y=mx+c. We know that the points of intersections of the circle and the straight line are the points which satisfy the simultaneous equation

$x^2+y^2=r^2$

$y = mx + c$ ... (2)

If we substitute mx + c for y in (1), we get

$x^2+(mx+c)^2-r^2=0$

which is same as the quadratic equation

$(1+m^2)x^2+2mcx+(c^2-r^2)=0$ ....(3)

This equation cannot have more than two solutions, and hence a line and a circle cannot intersect at more than two points. It is interesting to note that a substitution makes the problem of solving a system of two equations in two variables into a problem of solving a quadratic equation. Further we note that as the coefficients of the reduced quadratic polynomial are real, either both roots are real or both imaginary. If both roots are imaginary numbers, we conclude that the circle and the straight line do not intersect. In the case of real roots, either they are distinct or multiple roots of the polynomial. If they are distinct, substituting in (2), we get two values for y and hence two points of intersection. If we have equal roots, we say the straight line touches the circle as a tangent. As the polynomial (3) cannot have only one simple real root, a line cannot cut a circle at only one point.

17) Discuss the nature of the roots of the following polynomials:
$x^{2018}+1947x^{1950}+15x^8+26x^6+2019$

Let P(x) be the polynomial under consideration.

(i) The number of sign changes for P(x) and P(−x) are zero and hence it has no positive roots and no negative roots. Clearly zero is not a root. Thus the polynomial has no real roots and hence all roots of the polynomial are imaginary roots.

18) Discuss the nature of the roots of the following polynomials:
$x^5-19x^4+2x^3+5x^2+11$

Let P(x) be the polynomial under consideration.

The number of sign changes for P(x) and P(−x) are 2 and 1 respectively. Hence it has at most two positive roots and at most one negative root.Since the difference between number of sign changes in coefficients of P(−x) and the number of negative roots is even, we cannot have zero negative roots. So the number of negative roots is 1. Since the difference between number of sign changes in coefficient of P(x) and the number of positive roots must be even, we must have either zero or two positive roots. But as the sum of the coefficients is zero, 1 is a root. Thus we must have two and only two positive roots Obviously the other two roots are imaginary numbers.